Adaptive stabilization of unstable and nonminimum-phase stochastic systems

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Abstract: The system under consideration is the discrete time stochastic system $A(z)y_n = zB(z)u_n + w_n$, driven by a martingale difference sequence $\{w_n\}$, where $A(z)$ and $B(z)$ are polynomials in backward shift operator $z$ with unknown coefficients and both $A(z)$ and $B(z)$ may be unstable. With the purpose of demonstrating theoretical possibility rather than designing a practically applicable control law, this paper constructs an adaptive control that stabilizes the system and simultaneously guarantees strong consistency of the least squares estimates for unknown coefficients.

Keywords: Unstable system; nonminimum-phase stochastic system; adaptive stabilization; consistency.

1. Introduction

Consider the single-input single-output stochastic system

$$A(z)y_n = zB(z)u_n + w_n, \quad \forall n \geq 0, \quad y_n = u_n = 0, \quad \forall n < 0,$$

where $y_n$, $u_n$ and $w_n$ are the system output, input and unknown disturbance, respectively, and $A(z)$ and $B(z)$ are polynomials in backward shift operator $z$:

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \quad p \geq 0, \quad a_p \neq 0,$$

$$B(z) = b_1 + \cdots + b_q z^{q-1}, \quad q \geq 1, \quad b_q \neq 0.$$

Set

$$\theta = [-a_1 \ldots -a_p \ b_1 \ldots b_q]^T,$$

which is the unknown parameter of the system.

In adaptive control, to stabilize a nonminimum-phase and unstable system is a problem important in practice and appealing in theory. In the case where $w_n = 0 (\forall n \geq 0)$ in (1.1), this problem is considered in [16,7,12,1]. In the case where $w_n \neq 0$, this problem is the research topic of many papers, where, besides the standard coprimeness assumption on $A(z)$ and $zB(z)$, some additional conditions are usually required. For example, in [15,8,9,14] a lower bound of controllability (and observability) degree of the
systems is assumed to be known; in [6] it is required that a positive constant $\delta > 0$ is available so that \( \det[ B_n^T A_n B_n \cdots A_n^T B_n] > \delta > 0 \), where
\[
A_n = \begin{bmatrix}
a_{1n} & 1 \\
\vdots & \vdots \\
a_{sn} & 0
\end{bmatrix}, \\
B_n = \begin{bmatrix}
b_{1n} \\
\vdots \\
b_{sn}
\end{bmatrix}, \\
s = \max(p, q)
\]
and $\theta_n = [a_{1n} \cdots a_{pn} b_{1n} \cdots b_{qn}]^T$ is the estimate at time $n$ for unknown parameter $\theta$, $a_{in} = 0$, $b_{jn} = 0$ for $i > p$, $j > q$, while in [11, 2–5] stability is imposed either on $A(z)$ or on $B(z)$ when an adaptive control problem is solved, while controlling the system in the sense of adaptive $LQ$ or adaptive tracking.

It is worth mentioning that Giri, M’Saad, Dugard and Dion [10] have introduced a robust adaptive stabilization method for time-varying and ill-modelled systems with minimal priori knowledge. They assume that the system noise $\{w_t\}$ consists of unmodelled dynamics and uniformly bounded external disturbances.

In this paper, as in [10], except coprimeness we assume no additional condition on $A(z)$ and $B(z)$. For the case where $\limsup_{n \to \infty} n^{-1} \sum_{t=0}^{n} w_t^2 < \infty$, we give an adaptive control that stabilizes the system in the long run average sense, i.e.
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (u_i^2 + y_i^2) < \infty \quad \text{a.s.} 
\]
and simultaneously leads to strong consistency of the least squares estimate for $\theta$. Unlike [10], we do not assume that the external disturbance $\{w_t\}$ is uniformly bounded. For example, it may be a Gaussian white noise process. The key techniques used in the paper are ‘explosive excitation’ [13], ‘diminishing excitation’ [2] and ‘random truncations’ [3, 4]. The purpose of this work is to show the ability of adaptive control by exposing the minimum condition under which the system can adaptively be stabilized. To design a practically applicable control law is beyond the task we aim at.

2. Definition of adaptive control

In this paper, for a polynomial $X(z) = \sum_{i=0}^{\mu} x_i z^i$, the norm is defined as follows:
\[
\|X(z)\| = \left(\sum_{i=0}^{\mu} |x_i|^2 \right)^{1/2}.
\]
For estimating the unknown parameter $\theta$ we use the LS algorithm by which the estimate $\theta_n$ is recursively defined as follows:
\[
\theta_{n+1} = \theta_n + \mu_n P_n \phi_n (y_{n+1}^T - \phi_n^T \theta_n),
\]
\[
P_{n+1} = P_n - \mu_n P_n \phi_n \phi_n^T P_n, \\
\mu_n = (1 + \phi_n^T P_n \phi_n)^{-1},
\]
\[
\phi_n^T = [y_n \cdots y_{n-p+1} u_n \cdots u_{n-q+1}]
\]
with $P_0 = I$ and arbitrary initial value
\[
\theta_0^T = [-a_{10} \cdots -a_{p0} b_{10} \cdots b_{q0}].
\]
For any $n > 0$ write $\theta_n$ in the component form
\[
\theta_n^T = [-a_{1n} \cdots -a_{pn} b_{1n} \cdots b_{qn}].
\]
If $A(z)$ and $zB(z)$ are coprime, then there exist two polynomials
\begin{equation}
G(z) = 1 + \sum_{j=1}^{q-1} g_j z^j, \quad H(z) = \sum_{j=0}^{p-1} h_j z^j
\end{equation}
such that
\begin{equation}
A(z)G(z) - zB(z)H(z) = 1.
\end{equation}
Set
\begin{equation}
M_1 = \begin{pmatrix}
1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & a_1 & \cdots & \cdots & \cdots & a_p \\
0 & -b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & -b_1 & \cdots & \cdots & \cdots & -b_p \\
\end{pmatrix}
(\text{p + q by q}),
\end{equation}
\begin{equation}
M_2 = \begin{pmatrix}
1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & a_1 & \cdots & \cdots & \cdots & a_p \\
0 & -b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & -b_1 & \cdots & \cdots & \cdots & -b_p \\
\end{pmatrix}
(\text{p + q by p}),
\end{equation}
\begin{equation}
M = \begin{bmatrix}
M_1 & M_2
\end{bmatrix},
\end{equation}
\begin{equation}
\psi^T = \begin{bmatrix}
1 & g_1 & \cdots & g_{q-1} & h_0 & \cdots & h_{p-1}
\end{bmatrix},
\end{equation}
and
\begin{equation}
e^T = \begin{bmatrix}
1 & 0 & \cdots & 0
\end{bmatrix}_{1 \times (p + q)}.
\end{equation}
Replacing $a_i$, $b_i$, $g_k$, $h_k$ by their estimates $a_{i\alpha}$, $b_{i\alpha}$, $g_{k\alpha}$ and $h_{k\alpha}$ respectively in (1.2), (1.3), (2.5), (2.7)\textendash (2.10), $i = 1, \ldots, p$, $j = 1, \ldots, q$, $k = 1, \ldots, q - 1$, $s = 0, \ldots, p - 1$, we correspondingly denote $A(z)$, $B(z)$, $G(z)$, $H(z)$, $M_1$, $M_2$, $M$ and $\psi$ by $A_\alpha(z)$, $B_\alpha(z)$, $G_\alpha(z)$, $H_\alpha(z)$, $M_{1\alpha}$, $M_{2\alpha}$, $M_\alpha$ and $\psi_\alpha$ respectively; for example, $A_\alpha(z) = 1 + \sum_{i=1}^{p} a_{i\alpha} z^i$, $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$ and $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$ and $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$ and $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$ and $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$ and $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$ and $G_\alpha(z) = 1 + \sum_{j=1}^{q} g_{j\alpha} z^j$
We state the following simple fact as a lemma.

**Lemma 1.** If $A(z)$ and $zB(z)$ are coprime and $\theta_n \to \theta$ a.s. as $n \to \infty$, then there is an integer-valued $n_1$, possibly depending on sample path such that for any $n \geq n_1$ the equation
\begin{equation}
A_n(z)G_n(z) - zB_n(z)H_n(z) = 1
\end{equation}
has a unique solution $(G_n(z), H_n(z))$ satisfying
\begin{equation}
\deg(G_n(z)) \leq q - 1, \quad \deg(H_n(z)) \leq p - 1
\end{equation}
and
\begin{equation}
\|G_n(z)\|^2 + \|H_n(z)\|^2 \leq 1 + \|G(z)\|^2 + \|H(z)\|^2.
\end{equation}

**Proof.** It is easy to see that (2.6) has a unique solution $(G(z), H(z))$ if and only if the equation $M\psi = e$ has a unique solution or $\det(M) \neq 0$. Therefore, under the conditions of the lemma, $\det(M_n) \neq 0$ for sufficiently large $n$, and (2.10) has a unique solution with (2.13) satisfied.

Noticing that $\lim_{n \to \infty} n_\alpha = \lim_{n \to \infty} n_{\alpha} e^{-1} e = e^{-1} e = \psi$ we conclude that for any sample path (with possible exception of set of probability 0) there exists an integer $n_1$ such that for any $n \geq n_1$, (2.14) is fulfilled. \qed
From (2.6) it is clear that
\[
y_n = A(z)G(z)y_n - zB(z)H(z)y_n
= G(z)[A(z)y_n - zB(z)u_n] + zB(z)[G(z)u_n - H(z)y_n]
= G(z)w_n + zB(z)[G(z)u_n - H(z)y_n]
\]  
(2.15)
and
\[
u_n = H(z)w_n + A(z)[G(z)u_n - H(z)y_n].
\]  
(2.16)
Therefore, if \( \theta \) is known and \( u_n \) is defined from
\[
G(z)u_n - H(z)y_n = 0,
\]  
(2.17)
then by (2.15) and (2.16) we get \( y_n = G(z)w_n \) and \( u_n = H(z)w_n \). In this case, the system will be stabilized by controller (2.17) provided that \( \lim_{n \to \infty} n \sum_{i=1}^{n} w_i^2 < \infty \).

The 'certainty equivalence principle' suggests us to define adaptive control from
\[
G_\theta(z)u_n = H_\theta(z)y_n = 0.
\]  
(2.18)
However, in the present case the closeness of \( \theta_n \) to \( \theta \) is not guaranteed. Consequently, it is not clear if (2.12) is solvable or not. Even if \( G_\theta(z) \) and \( H_\theta(z) \) can be defined from (2.12) we still do not know whether or not they are close to \( G(z) \) and \( H(z) \) respectively. So it is important that \( \theta_n \) somehow approximates \( \theta \). If this is the case, then adaptive control defined by (2.18) may hopefully stabilize the system, and a stabilized system will in turn lead to a better estimate for \( \theta \) if the diminishing excitation technique is applied. For first step of approximating \( \theta \) we apply an explosive excitation input, by which we mean such an input that yields \( \lambda_{\min}^{(n)}/n \to \infty \) a.s. as \( n \to \infty \), where \( \lambda_{\min}^{(n)} \) denotes the minimum eigenvalue of \( P_{n+1}^{-1} = I + \sum_{i=0}^{n} \phi_i \phi_i^T \). To explain why an explosive excitation input will lead to an appropriate estimate for \( \theta \) we need the following lemma.

**Lemma 2.** Let \( \{w_n\} \) in (1.1) be any disturbance (deterministic or stochastic) satisfying the following condition:
\[
\lim_{n \to \infty} \sup_{n} \frac{1}{n} \sum_{i=1}^{n} w_i^2 < \infty.
\]  
(2.19)
Then the accuracy of the LS estimate \( \theta_n \) for \( \theta \) is expressed by
\[
\| \theta_n - \theta \|^2 = O(n/\lambda_{\min}^{(n)})
\]
where \( \lambda_{\min}^{(n)} \) denotes the minimum eigenvalue of \( P_{n+1}^{-1} = I + \sum_{i=0}^{n} \phi_i \phi_i^T \).

**Proof.** We will complete the proof with a similar argument used in Lemma 1 of [2]. Denote the estimation error by \( \tilde{\theta}_n \), i.e. \( \theta_n = \theta - \tilde{\theta}_n \). Then from (1.1), (2.1) and (2.2) it follows that
\[
\tilde{\theta}_{n+1} = \tilde{\theta}_n - P_k \phi_k (\tilde{\theta}_{n+1} \phi_k + w_{n+1}),
\]  
(2.20)
and
\[
P_{k+1}^{-1} = P_k^{-1} + \phi_k \phi_k^T.
\]
Thus, we have
\[
\tilde{\theta}_{k+1} = \tilde{\theta}_k + \left[ \tilde{\theta}_k \phi_k^T \right] P_k^{-1} \tilde{\theta}_k - 2(\tilde{\theta}_k \phi_k + w_{k+1}) \phi_k^T \phi_k + \phi_k^T P_k \phi_k [\tilde{\theta}_{k+1} \phi_k + w_{k+1}]^2.
\]  
(2.21)
By (2.20) we get
\[
\tilde{\theta}_k = \tilde{\theta}_{k+1} + P_k \phi_k (\tilde{\theta}_{k+1} \phi_k + w_{k+1}).
\]
Substituting this into the third term on the right hand side of (2.21) leads to
\[
\tilde{\theta}_{k+1}^T P_{k+1}^{-1} \tilde{\theta}_{k+1} = \left[ \tilde{\theta}_{k+1}^T \phi_k \right]^2 + \tilde{\theta}_{k+1}^T P_k \phi_k \left[ \tilde{\theta}_{k+1}^T \phi_k + \epsilon_{k+1} \right]^2 - 2 \left( \tilde{\theta}_{k+1}^T \phi_k + \epsilon_{k+1} \right) \tilde{\theta}_{k+1}^T P_k \phi_k \left[ \tilde{\theta}_{k+1}^T \phi_k + \epsilon_{k+1} \right] \phi_k
\]
\[- \tilde{\theta}_{k+1}^T P_k \phi_k - \left[ \tilde{\theta}_{k+1}^T \phi_k \right]^2 - 2 \epsilon_{k+1} \tilde{\theta}_{k+1}^T P_k \phi_k \left[ \tilde{\theta}_{k+1}^T \phi_k + \epsilon_{k+1} \right] \phi_k
\]
\[
\leq \tilde{\theta}_{k+1}^T P_k \phi_k - \left[ \tilde{\theta}_{k+1}^T \phi_k \right]^2 - 2 \epsilon_{k+1} \tilde{\theta}_{k+1}^T \phi_k.
\]

Summing up both sides and using (2.19) an noticing \(2 \mid ab \mid = 2(a / \sqrt{2} b / \sqrt{2}) \leq 1/2 a^2 + b^2\), we have
\[
\tilde{\theta}_{n+1}^T P_{n+1}^{-1} \tilde{\theta}_{n+1} = O(1) - \sum_{i=0}^{n} \left\| \tilde{\theta}_{i+1}^T \phi_i \right\|^2 - 2 \sum_{i=0}^{n} \epsilon_{i+1} \tilde{\theta}_{i+1}^T \phi_i
\]
\[
\leq O(1) - \frac{1}{2} \sum_{i=0}^{n} \left\| \tilde{\theta}_{i+1}^T \phi_i \right\|^2 + 2 \sum_{i=0}^{n} \epsilon_{i+1}^2 = O(n),
\]

which implies the desired result. \(\Box\)

Lemma 2 tells us that under an explosive excitation input the LS estimate is consistent. However, the stabilization purpose (1.5) does not allow us to apply such an input for a period longer than finite. Thus we need to define stopping times \(\sigma_i\), at which we turn off the explosive excitation input and switch on the control defined by the certainty equivalence principle until \(\tau_i\), at which the accuracy of the LS estimate \(\tilde{\theta}_n\) becomes unsatisfactory and we have to apply the explosive excitation input again. After defining stopping times
\[
0 =: \tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots
\]
it is most important to show that there is some integer \(i\) such that \(\sigma_i < \infty\) and \(\tau_i = \infty\), because otherwise the requirement (1.5) will never be met.

Let \(\{\epsilon_n\}\) be a real sequence with the following properties:
\[
0 < \epsilon_n < 1, \quad \epsilon_n \to 0, \quad \epsilon_n n^\alpha \to \infty \quad \text{for some } \alpha \in (0, \frac{1}{2}). \quad (2.22)
\]
(For example, \(\epsilon_n = [\log(10 + n)]^{-1}, \forall n > 0\).) We now define stopping times \(\sigma_i\) and \(\tau_i\) for any \(i > 1\):
\[
\sigma_i = \min \left\{ n > \tau_{i-1}; \sum_{j=0}^{n-1} \phi_j \phi_j^T - (\log s_n)^3 \epsilon_n^{-4} I > 0; \text{ det } M_n \neq 0 \right\};
\]
\[
\| G_n(z) \| ^2 + \| H_n(z) \| ^2 \leq \frac{1}{\max(\rho, q) \epsilon_n};
\]
\[
\left\| \frac{1}{s_n} \sum_{j=0}^{n-1} (y_j - \phi_j^T \theta_0) \right\|^2 \leq \frac{1}{s_n} \sum_{j=0}^{n-1} (y_j - \phi_j^T \theta_0) \leq 2 \epsilon_n^2 ,
\]
\[
\tau_i = \min \left\{ n > \sigma_i; \left\| \frac{1}{s_n} \sum_{j=0}^{n-1} (y_j - \phi_j^T \theta_{\sigma_j}) \right\|^2 \leq \frac{1}{s_n} \sum_{j=0}^{n-1} (y_j - \phi_j^T \theta_{\sigma_j}) \geq \epsilon_n^2 + \epsilon_n^2 \right\},
\]
\[
\quad (2.24)
\]
where \( s_n \) is given by \( s_0 = 1 \),
\[
s_n = n \max \left\{ 1, \frac{1}{k} \sum_{j=0}^{k-1} (y_j^2 + u_j^2), \ k = 1, \ldots, n \right\}, \ \forall n \geq 1.
\] (2.25)

Let \( \{v_n\} \) be independent of \( \{w_n\} \) and be mutually independent with the property
\[
E v_n = 0, \quad E v_n^2 = 1, \quad |v_n| < \sigma, \quad \forall n \geq 0, \quad v_n = 0, \quad \forall n < 0
\] (2.26)
with \( \sigma > 0 \) being a constant.

The explosive excitation signal \( \{u_n^e\} \) is defined from
\[
D(z) u_n^e = v_n, \quad u_0^e = 0,
\] (2.27)
where \( D(z) = 1 + \sum_{j=0}^{p+q} d_j z^j \) is an unstable polynomial of degree \( p + q \), i.e.
\[
D(z) \neq 0, \quad |z| > 1 \quad \text{and} \quad d_{p+q} \neq 0.
\] (2.28)

We now show that \( \{u_n^e\} \) really is explosive.

**Lemma 3.** If \( A(z) \) and \( zB(z) \) are coprime, \( \lim_{n \to \infty} n^{-1} \log \sum_{i=0}^{n} w_i^2 = 0 \) and \( u_n = u_n^e \), then
\[
\lim \inf_{n \to \infty} \frac{\lambda_{\min}(\|U_0 U_i^T\|)}{a^n} \geq c > 0 \quad a.s.
\] (2.29)
for some constant \( a > 1 \).

**Proof.** Replacing \( -b_i \) by \( b_i \), \( i = 1, \ldots, q \), in \( M_2 \) given by (2.8) we denote it by \( M_2' \). Set \( \Phi_n = A(z) \phi_n \) and \( D = [N_1^T M_1^T] \). From (1.1) it is easy to see that
\[
\Phi_n = DU_n + W_n,
\]
where \( U_n = [u_{n-1} \cdots u_{n-(p+q)+1}]^T \) and \( W_n = [w_{n-1} \cdots w_{n-(p+q)+1} 0 \cdots 0]^T \) (\( q \) zeros).

By coprimeness of \( A(z) \) and \( zB(z) \) we know that \( \det(D) \neq 0 \).

From (2.26)–(2.28) and Theorem 2 of [13] (see (A.2) in the Appendix) it follows that
\[
\lim \inf_{n \to \infty} \mu^{\lambda_{\min}(\|U_0 U_i^T\|)} \geq c > 0 \quad a.s.
\] (2.30)
where \( c \) possibly depends on sample path, \( \mu := \max(|z_i| : D(z_i) = 0, \ i = 1, \ldots, p + q < 1 \), and \( \lambda_{\min}(X) \) denotes the minimum eigenvalue of a matrix \( X \).

It is not difficult to see that for any \( \eta \in \mathbb{R}^{p+q} \) with \( \|\eta\| = 1 \),
\[
\sum_{i=0}^{n} \|\eta^T \Phi_i\|^2 \geq \frac{1}{2} \sum_{i=0}^{n} \|\eta^T DU_i\|^2 - \sum_{i=0}^{n} \|\eta^T W_i\|^2,
\]
which implies that
\[
\lambda_{\min}(\sum_{i=0}^{n} \Phi_i \Phi_i^T) \geq \frac{1}{2} \lambda_{\min}(\sum_{i=0}^{n} DU_i U_i^T) - \lambda_{\max}(\sum_{i=0}^{n} W_i W_i^T)
\]
\[
\geq \frac{1}{2} \lambda_{\min}(DD^T) \lambda_{\min}(\sum_{i=0}^{n} U_i U_i^T) - p \sum_{i=0}^{n} \|w_i\|^2.
\] (2.31)
On the other hand, noticing that \( \phi_i = 0 \) for any \( i < 0 \) we have

\[
\lambda_{\min} \left( \sum_{i=0}^{n} \phi_i \phi_i^T \right) - \inf_{\|x\|=1} \sum_{i=0}^{n} (x^T \phi_i)^2 = \inf_{\|x\|=1} \sum_{j=0}^{p} \left( x^T \phi_j + \sum_{j=1}^{p} a_j x^T \phi_{j-1} \right)^2 \\
\leq (p + 1) \left( 1 + \sum_{j=1}^{p} a_j^2 \right) \inf_{\|x\|=1} \sum_{i=0}^{n} (x^T \phi_i)^2 < (p + 1) \left( 1 + \sum_{j=1}^{p} a_j^2 \right) \lambda_{\min} \left( \sum_{i=0}^{n} \phi_i \phi_i^T \right).
\]

(2.32)

Combining (2.30)–(2.32), we find that \( \lambda_{\min}(\Sigma_{i=0}^{n} \phi_i \phi_i^T) \) diverges exponentially fast and (2.29) is true. \( \square \)

Finally, we define adaptive control \( u_n \) at time \( n \) is given by

\[
u_n = \begin{cases} 
\frac{u_n}{v_n} & \text{if } n \in [\tau_i, \sigma_{i+1}) \text{ for some } i \geq 0, \\
H_n(z) y_n - (G_n(z) - 1) u_n + \frac{v_n}{(n + 1)^2} & \text{if } n \in [\sigma_i, \tau_i) \text{ for some } i \geq 1,
\end{cases}
\]

(2.33)

where \( \epsilon \in [0, 1/4(p + q)) \).

We note that in the interval \( [\sigma_i, \tau_i) \) we add a diminishing signal \( v_n/(n + 1)^2 \) to the control defined by the certainty equivalence principle. The purpose of this is to make the LS estimate for \( \theta \) strongly consistent without damaging asymptotic behavior of the system. In contrast to [8,9,15] the upper bound of \( \|G(z)\|^2 + \|H(z)\|^2 \) is not assumed to be available in this paper. So in (2.23) we use a sequence \( \{1/\max(p, q) e_n\} \) diverging to infinity to dominate the estimated values of \( \|G_n(z)\|^2 + \|H_n(z)\|^2 \).

3. Main results

For convenience we formulate some known results on LS estimates as Theorem 1. For its proof we refer to [2,13].

**Theorem 1.** If \( A(z) \) and \( zB(z) \) are coprime, \( (w_n, \mathcal{F}_n) \) is a martingale difference sequence with

\[
sup_{n \geq 0} E \left[ w_{n+1}^2 | \mathcal{F}_n \right] < \infty \text{ and } \sum_{i=0}^{n} w_i^2 = O(n),
\]

(3.1)

and \( u_n \) is \( \mathcal{F}_n \)-measurable, then for any \( \gamma > 1, \)

\[
\sum_{j=0}^{n-1} \xi_j^2 = O(\log r_n(\log \log r_n)^\gamma), \quad \|\theta_n - \theta\|^2 = O \left( \frac{\log r_n(\log \log r_n)^\gamma}{\lambda_{\min}^{(n-1)}} \right),
\]

(3.2)

where

\[
\xi_n = y_n - \phi_{n-1}^T \theta_n - w_n, \quad r_n = 1 + \sum_{j=0}^{n-1} \|\phi_j\|^2,
\]

(3.3)

and \( \{\xi_n\} \) is a family of nondecreasing \( \sigma \)-algebras.

Further, if, in addition, \( u_n = u_n^T + v_n/(n + 1)^2, \) \( \epsilon \in [0, 1/4(p + q)) \), where \( v_n \) is given by (2.26) and \( u_n^T \) is measurable with respect to \( \mathcal{F}_{n-1} \) and \( \Sigma_{j=1}^{n-1}(u_j)^2 = O(n) \), and if the system output satisfies \( \Sigma_{i=1}^{n}(\phi_i)^2 = O(n) \), and if the system output satisfies \( \Sigma_{i=1}^{n}(\phi_i)^2 = O(n) \), then

\[
\|\theta_n - \theta\|^2 = O \left( \frac{\log n(\log \log n)^\gamma}{n^{1-2\epsilon(p+q)}} \right), \quad \forall \gamma > 1.
\]

(3.4)
Theorem 2. If \( A(z) \) and \( zB(z) \) are coprime, \( \{w_n, \mathcal{F}_n\} \) is a martingale difference sequence with (3.1) satisfied, then the adaptive control (2.35) stabilizes the closed-loop system (1.1) and the 1S estimate \( \theta_n \) given by (2.1)–(2.3) is strongly consistent, precisely,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left( y_j^2 + u_j^2 \right) < \infty \quad \text{a.s.,} \\
\| \theta_n - \theta \|^2 = O\left( \frac{(\log n)(\log \log n)^\gamma}{n^{1-2\gamma(\rho+\delta)}} \right) \quad \text{for any } \gamma > 1.
\]

Proof. The first step is to show that there exists an integer \( i \) such that \( \sigma_i < \infty \) and \( \tau_i = \infty \).

By (2.25) it is clear that \( s_n/n \) is nondecreasing,

\[
s_n = O(nr_n) \quad \text{and} \quad r_n = O(s_n). \quad (3.7)
\]

We now show that it is impossible that \( \tau_i < \infty \) and \( \sigma_{i+1} = \infty \) for some \( i \geq 0 \). If it were not true, i.e. if there were an \( i \geq 0 \) such that \( \tau_i < \infty \) and \( \sigma_{i+1} = \infty \), then \( u_n = u_n^i, \forall n \geq \tau_i \). Noticing that \( \{v_n\} \) is bounded, it is easy to see that there exists \( b > 1 \) such that

\[
r_n = O(b^n), \quad (3.8)
\]

which together with (3.7) leads to \( \log s_n = O(n) \). Thus, from (2.22) and (2.29) it follows that for some integer \( N_0 \geq 0 \) and all \( n \geq N_0 \), the first inequality in the definition (2.23) is true.

By Lemmas 2 and 3,

\[
\| \theta_n - \theta \|^2 = O(n/a^n) \quad \text{a.s.} \quad (3.9)
\]

From this and Lemma 1 we see that for some integer \( N_i \geq N_0 \geq 0 \) and all \( n \geq N_i \), the second and third inequalities in the definition (2.23) hold.

By (1.1), (3.1)–(3.3), (3.7)–(3.9) and Schwarz inequality we have

\[
\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi^T_{j-1} \theta_n \right)^2 - \frac{1}{sn} \sum_{j=0}^{n-1} \left( y_j - \phi^T_{j-1} \theta_j \right)^2 \right|
\leq 2 \frac{1}{s_n} \sum_{j=0}^{n-1} |w_j| \left| \phi^T_{j-1} \theta - \phi^T_{j-1} \theta_n \right| + \frac{1}{s_n} \sum_{j=0}^{n-1} \left( \phi^T_{j-1} \theta - \phi^T_{j-1} \theta_n \right)^2
\leq 2 \frac{1}{s_n} \sum_{j=0}^{n-1} |w_j| \sum_{j=0}^{n-1} |\xi_j|^2 \quad \text{by (1.1) and (3.3)}
\leq 2 \frac{1}{s_n} \left( \sum_{j=0}^{n-1} |w_j|^2 \right)^{1/2} \left( \sum_{j=0}^{n-1} |\xi_j|^2 \right)^{1/2}
\leq 2 \frac{1}{s_n} \left( \sum_{j=0}^{n-1} |w_j|^2 \right)^{1/2} \left( \sum_{j=0}^{n-1} |\xi_j|^2 \right)^{1/2} \quad \text{(by Schwarz inequality)}
\leq \frac{1}{a^n} \left( \sum_{j=0}^{n} y_j^2 \right) \quad \forall \gamma > 1 \quad \text{(by (3.1), (3.2), (3.7)–(3.9)).} \quad (3.10)
\]

From this and (2.22) it follows that there exists an integer \( N_2 \geq N_i \geq 0 \) such that for any \( n \geq N_2 \),

\[
\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi^T_{j-1} \theta_n \right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi^T_{j-1} \theta_j \right)^2 \right| \leq 2 \epsilon_n^2,
\]

i.e. the last inequality in the definition (2.23) is true.
Therefore, we have \( \sigma_{i+1} \leq N_2 \). This contradicts \( \sigma_{i+1} = \infty \).

We now prove that \( \tau_i = \infty \) for some \( i \). By (3.2) it follows that

\[
\| \theta_{\sigma_i} - \theta \|^2 = O\left( (\log s_{\sigma_i})^2 / s_{\min}^{(n-1)} \right),
\]

which incorporating the definition of \( \sigma_i \) implies that

\[
\| \theta_{\sigma_i} - \theta \|^2 = O\left( \varepsilon_{\sigma_i}^2 \right) .
\]  

(3.12)

Similar to (3.10), by (3.2) we obtain that

\[
\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi_{j-1}^T \theta_{\sigma_i} \right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi_{j-1}^T \theta \right)^2 \right| = O\left( \frac{\| \theta_{\sigma_i} - \theta \| + \log s_n}{s_n} \right).
\]  

(3.13)

By (3.12), (3.13) and \( s_n \gg n \) we know that there exists and \( i_0 \) such that for any \( i \geq i_0 \) and any \( n \gg \sigma_i \),

\[
\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi_{j-1}^T \theta_{\sigma_i} \right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left( y_j - \phi_{j-1}^T \theta \right)^2 \right| \leq \varepsilon_{\sigma_i}^2 + \varepsilon_n^2 .
\]

Thus \( \tau_{i_0} = \infty \). For simplicity of notations we write \( i_0 \) as \( i \), i.e. \( \sigma_i < \infty \), \( \tau_i = \infty \) a.s.

By (2.12) we have

\[
y_n = G_{\sigma}(z) \left[ A_{\sigma}(z) y_n - zB_{\sigma}(z) u_n \right] + zB_{\sigma}(z) \left[ G_{\sigma}(z) u_n - H_{\sigma}(z) y_n \right],
\]

\[
u_n = H_{\sigma}(z) \left[ A_{\sigma}(z) y_n - zB_{\sigma}(z) u_n \right] + A_{\sigma}(z) \left[ G_{\sigma}(z) u_n - H_{\sigma}(z) y_n \right].
\]

(3.14)

(3.15)

Hence, from (3.14), (3.15) and (2.33) we get, for any \( n \gg n_0 \) := \( \sigma_i + \max(\rho, q) \),

\[
y_n = G_{\sigma}(z) \left[ A_{\sigma}(z) y_n - zB_{\sigma}(z) u_n \right] + zB_{\sigma}(z) \frac{\nu_n}{(n+1)} ,
\]

\[
u_n = H_{\sigma}(z) \left[ A_{\sigma}(z) y_n - zB_{\sigma}(z) u_n \right] + A_{\sigma}(z) \frac{\nu_n}{(n+1)} .
\]

(3.16)

(3.17)

By noticing \( |v_n| \leq \sigma \) and the elementary inequality \( \sigma^2 (1 + \beta^2) \leq (1 + \alpha^2 + (1 + \nu)/\nu) \beta^2 \), \( \forall \nu > 0 \), \( \forall \alpha \), \( \forall \beta \), from (3.16) and (3.17) it follows that for any \( n \gg n_0 \),

\[
\frac{1}{n} \sum_{j=0}^{n-1} (y_j^2 + u_j^2) = \frac{1}{n} \sum_{j=0}^{n-1} (y_j^2 + u_j^2) + \frac{1}{n} \sum_{j=0}^{n-1} (y_j^2 + u_j^2)
\]

\[
\leq \frac{1 + \nu}{n} \max(\rho, q) \left( \| G_{\sigma}(z) \|^2 + \| H_{\sigma}(z) \|^2 \right) \sum_{j=0}^{n-1} (y_j - \phi_{j-1}^T \theta_{\sigma_i})^2 + \frac{1 + \nu}{\nu n} \left( 1 + \max(\rho, q) \right) \left( \| A_{\sigma}(z) \|^2 + \| B_{\sigma}(z) \|^2 \right) \sum_{j=1}^{n-1} \frac{v_j^2}{(j+1)^2 e} + \frac{1}{n} \sum_{j=0}^{n-1} (y_j^2 + u_j^2) \quad \text{(by (3.16) and (3.17))}
\]
\[\frac{1 + \nu}{\varepsilon_{\alpha_i}} \frac{1}{n} \sum_{j=0}^{n-1} (y_j - \bar{y}^T \theta_{\alpha_i})^2 + O(1) \quad (\text{by (2.23) and } |v_j| < \alpha)\]
\[\leq \frac{1 + \nu}{\varepsilon_{\alpha_i}} \frac{1}{s_n} \sum_{j=0}^{n-1} (y_j - \bar{y}^T \theta_{\alpha_i})^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} (y_j - \bar{y}^T \theta_{\alpha_j})^2\]
\[+ \frac{1 + \nu}{\varepsilon_{\alpha_i}} \frac{1}{n} \sum_{j=0}^{n-1} (y_j - \bar{y}^T \theta_{\alpha_j})^2 + O(1)\]
\[\leq \left(1 + \nu\right)\left(\varepsilon_{\alpha_i} + \frac{\varepsilon_n^2}{\varepsilon_{\alpha_i}}\right) \frac{s_n}{n} + O\left(\frac{\log s_n}{n}\right) + O(1),\]  
(3.18)

where for the last inequality we have used (2.24) and \( \tau_i = \infty. \)

Since \( \varepsilon_{\alpha_i} < 1 \) and \( \varepsilon_n \to 0, \) we can take \( \nu > 0 \) small enough and \( n_1 \) sufficiently large so that \( (1 + \nu)(\varepsilon_{\alpha_i} + \varepsilon_n^2/\varepsilon_{\alpha_i}) \leq \beta < 1 \) for all \( n \geq n_1. \) Therefore, for any \( n \geq n_1, \)

\[\frac{1}{n} \sum_{j=0}^{n-1} \left(y_j^2 + u_j^2\right) \leq \frac{s_n}{n} + c_1,\]  
(3.19)

where \( c_1 < \infty \) is independent of \( n. \)

Noticing that \( s_n/n \) is nondecreasing from (3.19) we get for any \( n \geq n_1 \) and any \( l \in [n_1, n], \)

\[\frac{1}{l} \sum_{j=0}^{l-1} \left(y_j^2 + u_j^2\right) \leq \frac{s_n}{l} + c_1 \leq \frac{s_n}{n} + c_1,\]

which together with (2.25) yields

\[\frac{s_n}{n} \leq \max\left\{1; \frac{1}{l} \sum_{j=0}^{l-1} \left(y_j^2 + u_j^2\right), l = 1, \ldots, n_1 - 1; \beta \frac{s_n}{n} + c_1\right\},\]

or

\[\frac{s_n}{n} \leq \beta \frac{s_n}{n} + c_2\]  
(3.20)

where

\[c_2 = 1 + c_1 + \max\left\{\frac{1}{l} \sum_{j=0}^{l-1} \left(y_j^2 + u_j^2\right), l = 1, \ldots, n_1 - 1\right\} .\]

Consequently, \( s_n/n \leq (1 - \beta)^{-1} c_2, \) i.e., (3.5) follows, while (3.6) implied by Theorem 1. \( \Box \)

4. Conclusion

For the stochastic system, which possibly is of nonminimum-phase and open-loop unstable, we basically use the adaptive control constructed by the certainty equivalence principle and excited by a diminishing signal, but with explosive inputs applied at some random time intervals of finite length. As results, both adaptive stabilization and consistent parameter estimate are achieved.
Appendix

Consider the autoregressive AR(p) model

\[ y_n = \beta_1 y_{n-1} + \cdots + \beta_p y_{n-p} + \varepsilon_n, \quad (A.1) \]

where \( y_n \) is the observation and \( \varepsilon_n \) is the random disturbance at stage \( n \), and \( \beta_1, \ldots, \beta_p \) are the parameters of the model.

Let

\[ f(z) = z^p - \beta_1 z^{p-1} - \cdots - \beta_p, \]

\[ Y_n = [y_n, \ldots, y_{n-p+1}]^T, \quad L_n = [\varepsilon_n, 0, \ldots, 0]^T \quad (p - 1 \text{ zeros}) \]

\[ F = \begin{pmatrix} \beta_1, \ldots, \beta_{p-1} & \beta_p \\ I_{p-1} & 0 \end{pmatrix} \quad \text{and} \quad X_n = [Y_p, Y_{p+1}, \ldots, Y_{n-1}]^T. \]

Then Lai and Wei [13] obtain the following results.

**Theorem A.** Suppose that in the AR(p) model (A.1), \( \{\varepsilon_n\} \) is a martingale difference sequence with respect to an increasing sequence of \( \sigma \)-fields \( \{\mathcal{F}_n\} \) such that

\[ \liminf_{n \to \infty} E(\varepsilon_n^2 | \mathcal{F}_{n-1}) > 0 \quad \text{a.s.} \]

Assume that the roots \( z_j \) of the characteristic polynomial \( f(z) \) lie outside the unit circle, i.e., \( |z_j| > 1 \) for \( j = 1, \ldots, p \). Let \( \bar{m} = \min_{1 \leq j \leq p} |z_j| \) and \( \bar{M} = \max_{1 \leq j \leq p} |z_j| \). Then we have

(i) \( F^{-1} Y_p \) converges a.s. to \( Z := Y_p + \sum_{i=1}^{p} F^{-1} D_{i-p} \). Moreover, \( x^T Z \) has a continuous distribution for all \( x \in \mathbb{R}^p \).

(ii) \( F^{-1} X_n^T X_n (F^{-1})^T \) converges a.s. to \( \Sigma := \sum_{i=p+1}^{\infty} F^{-1} (Z^T Z) (F^{-1})^T \). Moreover, \( \Sigma \) is positive definite with probability 1. Consequently,

\[ \lim_{n \to \infty} n^{-1} \log \lambda_{\min}(X_n^T X_n) = 2 \log \bar{m} \quad \text{a.s.,} \quad (A.2) \]

\[ \lim_{n \to \infty} n^{-1} \log \lambda_{\max}(X_n^T X_n) = 2 \log \bar{M} \quad \text{a.s.} \quad (A.3) \]

References


